Dense loops, supersymmetry, and Goldstone phases in two dimensions

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(Date: February 1, 2008)

Loop models in two dimensions can be related to O(N) models. The low-temperature dense-loops phase of such a model, or of its reformulation using a supergroup as symmetry, can have a Goldstone broken-symmetry phase for $N < 2$. We argue that this phase is generic for $-2 < N < 2$ when crossings of loops are allowed, and distinct from the model of non-crossing dense loops first studied by Nienhuis [Phys. Rev. Lett. **49**, 1062 (1982)]. Our arguments are supported by our numerical results, and by a lattice model solved exactly by Martins *et al.* [Phys. Rev. Lett. **81**, 504 (1998)].

It has long been understood that theories of an N-component scalar field ϕ with O(N) symmetry in Euclidean space of dimension d can be related to statistical-mechanics models in which configurations of loops are given Boltzmann weights depending on their lengths and intersection properties [1]. In the $N \to 0$ limit, unwanted closed loops vanish [1], and the usual self-avoiding walks (polymers) are related to the generic critical point in the O($N \to 0$) theory. The $N \to 0$ limit can be avoided by using a model with a Lie superalgebra as symmetry ("supersymmetry") [2]. Other values of N are also of interest. In $d = 2$, exact results based on lattice models [3] show that there also exists a massless low-temperature (low-T) regime of the model, for $-2 < N \leq 2$. The properties of this phase—in which, geometrically, the loops become dense—are well-understood using Coulomb gas techniques [3].

In this Letter, we revisit the low-T phase of the O($N \leq 2$) models in $d = 2$. The supersymmetry approach, which can be generalized to all integer N, makes it clear that for $N < 2$ there should be a massless phase described by the Goldstone modes of the broken O(N) or supersymmetry, with very simple scaling dimensions. We argue that the usual dense-loops phase [3], in which loops never intersect, is not this Goldstone phase (in contrast to earlier expectations [4]): in fact, it is not generic, and an arbitrarily-weak perturbation that allows loop intersections causes a cross-over to the generic Goldstone phase for $-2 < N < 2$. We present numerical results that support our interpretation. There is also a soluble model that appears to be in the Goldstone phase [5].

The generic continuum O(N)-invariant action S for the scalar field ϕ includes the interaction term $-\lambda(\phi\cdot\phi)^2$ (by convention the Boltzmann weight is e^S). In terms of loops, this is the model introduced by de Gennes [1], in which the loops can cross, but for $\lambda > 0$ these crossings are disfavored. It has a second-order transition for $d > 2$, but for $d = 2$, a transition occurs only for $N \leq 2$; for $N > 2$, there is only the high-T or massive unbroken-symmetry phase. (For N not a positive integer, the theory is supposed to be defined by analytic continuation of its perturbation expansion. For N integer, this can be made rigorous by the use of supersymmetry, as we describe below.) When $-2 \leq N \leq 2$, the phase transition is second order, and the associated critical exponents have been conjectured by Nienhuis [3] and others [6]. For $N < -2$, the transition is expected to be first order.

The existence of the transition for $N \leq 2$ implies the existence of a low-T phase. From the point of view of the generic O(N) ϕ^4 theory, at low T the symmetry is broken to O($N-1$) according to Landau theory, and the low-T phase would be expected to be a Goldstone phase with massless excitations, described by a nonlinear sigma model with target manifold O(N)/O($N-1$) $\cong S^{N-1}$, a sphere. For $N = 2$, the O(2) symmetry is not really broken, but there is a power-law phase with continuously-varying exponents, the Berezinskii-Kosterlitz-Thouless (BKT) phase [7]. For $N < 2$, use of the perturbative beta function in the S^{N-1} sigma model, defined by analytic continuation in N, implies that the coupling between the Goldstone modes in the sigma model becomes weak at large length scales [8], like Goldstone phases in higher d. This yields a low-T phase with simple forms for the scaling dimensions. This is possible in $d = 2$ for $N < 2$ because the Mermin-Wagner theorem no longer applies when the continuous SO(N) symmetry cannot be realized as unitary operations on a vector field. We discuss this theory more fully below, using supersymmetry.

In contrast, Refs. [9,3] defined a particular lattice model of strictly non-crossing loops. This was done by truncating the high-T expansion of a lattice version of the ϕ^4 theory as above [9], in order to simplify calculations of critical properties while hopefully remaining in the same universality class. The model has Boltzmann weight

$$e^S \equiv \prod_{\langle ij\rangle} \left(1 + K\vec{S}_i\cdot\vec{S}_j\right), \tag{1}$$

where i, j are vertices on the honeycomb graph, $\langle ij\rangle$ denotes an edge of the graph, and the variables $\vec{S}_i$ are N-component fixed-length spins, $\vec{S}_i^2 = N$. The partition function can be evaluated as $Z = \sum N^L K^E$ where the sum is over graphs consisting of self-avoiding mutually-avoiding loops, L is the number of loops, E the number of edges they occupy, and $K \sim 1/T$. The critical K for

this model is known exactly, $K_c = \left(2+\sqrt{2-N}\right)^{-1/2}$ for $N \leq 2$ [3]. The transition is second-order for $-2 \leq N \leq 2$ only, and the critical exponents at this critical point, which is known as *dilute loops* (or as self-avoiding walks for $N \to 0$), are known by a variety of techniques [3,6]. The low-T region for each $-2 < N < 2$ is attracted to a unique massless phase (that varies with N) for all $K_c < K < \infty$. This phase has nontrivial scaling dimensions, which are again known [3,10]. It has become known as *dense loops* (or as dense polymers for $N \to 0$). In the limit $N \to 2$, this phase coincides with the critical end-point of the BKT phase (which appears in the $N = 2$ model for $K \geq K_c$). For $N < -2$, we expect that the dense-loops phase is massive.

Both the critical and dense-loops phases are believed to be universal as continuum phases, in the sense that they are independent of the choice of lattice used in defining them. Recent work [11] on the $T = 0$ limits of certain lattice models identifies other massless phases of non-intersecting *fully-packed* loops, that depend on the lattice used. We are not concerned with these here, nor with the transition for $N > 2$ in the region $|K| > 1/N$, where the Boltzmann weight (1) can be negative [12].

From the critical exponents for dilute loops, it is natural to hope [3], that they are in the same universality class as the transition in the generic $\mathrm{O}(N)$ ϕ^4 theory. The idea is that any weak repulsion of the lines leads to a crossover to the infinitely-repulsive limit in which the loops never cross, so that the latter fixed point governs the transition for all $\lambda > 0$. We know of no reason to doubt this. As a check, we can reintroduce intersections of the lines in the dilute critical theory. The graphs of the expansion of the generic $\mathrm{O}(N)$ model that are not present in the truncated model would involve in particular crossings of the loops, or multiple occupancies of edges. In the continuum description, the most relevant (and most generic) of these is where two lines cross. This may be included as a perturbation of the dilute critical theory by adding to the action the integral of a multiple of the corresponding scalar operator, the so-called 4-leg operator, which has conformal weight [6] $h_4 = g - (g-1)^2/(4g)$, where we parameterized $N = -2\cos\pi g$, $g \in [1, 2]$. We see that this operator is marginal at $N = 2$ and irrelevant (i.e. $h_4 > 1$) for $-2 \leq N < 2$. This strongly suggests that allowing more general interactions or configurations of loops in the lattice model would not change the universality class, at least when the perturbations are small, and hence that the *dilute* critical point [3] is indeed generic.

The model of non-intersecting loops was obtained by truncating the high-T expansion, and this would be expected not to change the behavior for sufficiently high T (small K); apparently this remains true down to $T = T_c$. However, this leaves open the possibility that the low-T behavior of the models is different, which we will now examine. The dense-loops phase is described by a theory similar to that of the dilute case, with the same parameterization of N and h_4 as above, but with $g \in [0, 1]$. One sees that the 4-leg operator is now relevant in the whole region $-2 < N < 2$. As $N \searrow -2$, $h_4 \searrow -\infty$. Since the 4-leg perturbation is relevant in the low-T phase, it is *dangerously* irrelevant in the critical theory.

It is thus clear that the *dense*-loops phase is non-generic from an $\mathrm{O}(N)$-model point of view. In $d > 2$, de Gennes' model appears generic, as the topological effect of strict non-intersection, present in $d = 2$, is absent (though in $d = 3$, there remains the topology of knots and linking instead). Even a model in $d > 2$ of strict non-intersection, when reduced to $d = 2$ by confining $d - 2$ dimensions to finite intervals, gives an effective model in which intersections still occur. There is also a symmetry interpretation for the non-genericity: the dense phase of strictly non-intersecting loops has a larger symmetry, $\mathrm{U}(N)$ instead of $\mathrm{O}(N)$, which is based on the possibility of consistently orienting all the loops in the partition sum [8,3]. Though this symmetry may be broken by the boundary conditions, the massless dense-loops phase appears to be most easily understood in terms of a non-linear sigma model with this larger symmetry [8]. The 4-leg operator is a symmetry-breaking perturbation, that reduces the symmetry to the generic $\mathrm{O}(N)$. It would be surprising if the large-length-scale effect of this relevant symmetry-breaking perturbation led back to the same higher-symmetry phase. We note that the importance of the self-crossings has been observed in the related context of Lorentz lattice gases [13].

We now argue that the flow induced by the 4-leg perturbation leads to the generic Goldstone phase of the ϕ^4 theory, and begin by examining the structure of this phase, before giving numerical evidence that such a flow does in fact occur.

The loop (high-T) expansion of the $\mathrm{O}(N)$ model can be reproduced by considering instead a model with $\mathrm{OSp}(m|2n)$ symmetry, with $N = m - 2n$; in particular, each closed loop incurs a factor of N [2,8]. The model then makes complete sense even for N non-positive, (but N must be an integer). The phase diagram, beta functions, and scaling dimensions are the same for all m at fixed N (though multiplicities of operators may vanish for m small [8]). The Goldstone phase should thus be described by a sigma model with target space $\mathrm{OSp}(m|2n)/\mathrm{OSp}(m-1|2n) \cong S^{m-1|2n}$, a supersphere, and a single coupling constant g_σ [8,14]. The perturbative beta function is, to leading order, $\beta \equiv dg_\sigma/d\ln L \propto (N-2)g_\sigma^2$, so for $N < 2$, the model flows to weak coupling for $g_\sigma \geq 0$ (which is the expected physical sign). Again, the Mermin-Wagner theorem does not apply here, and symmetry-breaking is allowed for $N < 2$. The weak-coupling fixed point is a theory of free massless scalars, with $m - 1$ bosonic and $2n$ fermionic components, so is conformal with central charge $c = m-1-2n = N-1$, as it would be for S^{N-1}. There will be logarithmic corrections

due to the marginally-irrelevant coupling g_σ. We emphasize that in the sigma model, $\mathrm{OSp}(m|2n)$ symmetry does not allow any other marginal or relevant couplings to be added to the action, so that this fixed point is robust, unlike the dense-loops theory above.

Explicitly, the target manifold can be parameterized by commuting coordinates x_i, $i = 1, \ldots, m$, and anti-commuting coordinates η_j, $j = 1, \ldots, 2n$, subject to the constraint $\sum_{i=1}^m x_i^2 + \sum_{j=1}^n \eta_{2j-1}\eta_{2j} = 1$. The action of the sigma model is

$$S = -\frac{1}{g_\sigma} \int d^2r \left[\sum_{i=1}^m (\partial_\mu x_i)^2 + \sum_{j=1}^n \partial_\mu \eta_{2j-1} \partial_\mu \eta_{2j} \right]. \quad (2)$$

The simplest example is $m = n = 1$ ($N = -1$), which corresponds to the $S^{0|2}$ supersphere. The constraint can be solved as $x_1 = 1 - \frac{1}{2}\eta_1\eta_2$. After elimination of x_1 and some rescaling, one finds

$$S = -\int d^2r \left[\partial_\mu \eta_1 \partial_\mu \eta_2 - \frac{g_\sigma}{2} \eta_1 \eta_2 \partial_\mu \eta_1 \partial_\mu \eta_2 \right]. \quad (3)$$

At long wavelengths the coupling g_σ renormalizes towards zero, and the theory becomes free symplectic fermions, with central charge $c = -2$, as claimed. In general, for $N < 1$, the partition function of the fixed-point theory vanishes, $Z = 0$. Similar behavior is well-known in the dense-polymer ($N = 0$) phase.

The supersymmetry analysis is valid for N integer. For general N, we are forced again to interpolate formally. The availability of the supersymmetry results greatly strengthens our confidence in this procedure and in the existence of the Goldstone phase for all $N < 2$.

Both at the dilute critical point and in the dense-loops phase, the central charge is $c = 1 - 6(g-1)^2/g$, in the parameterization above. For dense loops, $g \in [0, 1]$, this coincides with $c = N - 1$ for $N = 1, 2$. The $N = 2$ case is the BKT phase with $c = 1$ throughout, and not of interest here. We claim that, for all $-2 < N < 2$, the perturbed dense-loops phase flows to the Goldstone phase, which is a distinct massless phase even when it has the same c. For $1 < N < 2$, c decreases during the flow, but for $-2 < N < 1$ it increases. This latter behavior would not violate Zamolodchikov's c-theorem, as the theories involved are non-unitary. In particular, the $N = 0$ ($c = -2$) dense-polymer theory should flow to a $c = -1$ free theory. Note that for $N = 1$ the partition function is $Z = 1$ (after removing nonuniversal constants) in both the dense-loops and Goldstone phases, a consequence of $c = 0$.

Some particularly interesting operators in loop models in general are the k-leg operators (also known as fuseaux or watermelon operators), insertions of which give the probability that k lines terminate at the same point. In the general $\mathrm{O}(N)$ point of view, the leading such operator would be represented by k insertions of the field ϕ at the same point (with no derivatives), and must necessarily be in a totally-symmetric rank-k tensor representation of $\mathrm{O}(N)$, which can be assumed to be traceless on all pairs of indices. For the $\mathrm{OSp}(m|2n)$ formulation, this becomes a supersymmetric tensor, that vanishes when contracted using the same invariant bilinear form as in the constraint in the sigma model. It is useful to choose the index values of the set of tensors in a correlation function to occur in distinct pairs, to force the lines in the loop model to connect the positions of the operators in some specified way. This is possible if m is sufficiently large, a typical situation in applications of supersymmetry.

At the dilute critical point and in the dense-loops phase, there are nontrivial scaling dimensions X_k (or conformal weights $h_k = X_k/2$) for the k-leg operators, which are known exactly [6,10] (for example, h_4 was given earlier). (In these phases, the symmetric-tensor k-leg operators are in general degenerate with others [8], and in the dense-loops phase, we assume there is a singlet 4-leg operator, which we used as the perturbation above; it is the most relevant invariant scalar operator.) In the Goldstone phases, on the other hand, the symmetric tensors correspond to functions like spherical harmonics on the target space, and their scaling dimensions tend to zero as $g_\sigma \to 0$. Since the symmetry is broken for $N < 2$, the precise form of each operator depends on which components are chosen. If the expectation of the field is $x_1 = 1$, all other x_i and $\eta_j = 0$, then any component of the k-leg operator can be rewritten in terms of the (rescaled, free) $m-1$ x_i's and the $2n$ η_j's, and the correlation functions will contain various logarithmic factors depending on which components are chosen. It is clear that the scaling dimensions of the leading k-leg operators are all zero, and there will be subleading operators with integer conformal weights (the precise form of correlation functions may require more attention to the limit $g_\sigma \to 0$). When the dense-loops phase is perturbed by the 4-leg operator, it should be possible to see the effective scaling dimensions of the k-leg operators (say, in two-point functions) cross over to zero, for all k.

Numerical transfer matrix calculations were done on a square-lattice cylinder of circumference L. The edges of the lattice are packed with one line per edge; two lines can meet at each vertex in three ways, the weight being 1 for a non-crossing and w for a crossing. The natural $\mathrm{O}(N)$-invariant boundary condition for a periodic system is that the fields ϕ are periodic, both for Bose and Fermi components. This implies that a loop that winds the periodic direction is given the same factor N as a topologically trivial loop. In a conformally-invariant system, c can be extracted from the scaling of the free energy per vertex, $f(L) = f(\infty) - \pi c/(6L^2)$. We therefore use this formula to extract the effective value of c in a crossover, using $f(L)$ at pairs of values of L. Results for $N = 0$ (see Fig. 1) show a clear trend away from $c = -2$, and are consistent with a slow approach to $c = -1$ for $L \to \infty$.

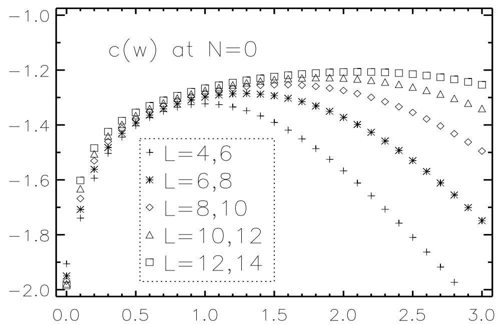

FIG. 1. Measures of the central charge vs. w for $N = 0$. The conjectured value for $L \to \infty$ is $c = -1$ for all $w > 0$.

For the k-leg scaling dimensions, we use the lowest transfer matrix eigenvalues E_k in each sector with k lines propagating along the cylinder. From conformal invariance [15], we define effective values of X_k at finite L from $E_k - E_0 = 2\pi X_k/L$; see Fig. 2 ($k = 2$ is not included as, for $N = 0$ in our model, $E_2 = E_0$ for all L and w). The results are consistent with $X_k = 0$ in the fixed-point theory. In a situation with a marginally-irrelevant operator, one expects [15] corrections to scaling to be $O(1/L \ln L)$, which presumably explains the slow convergence.

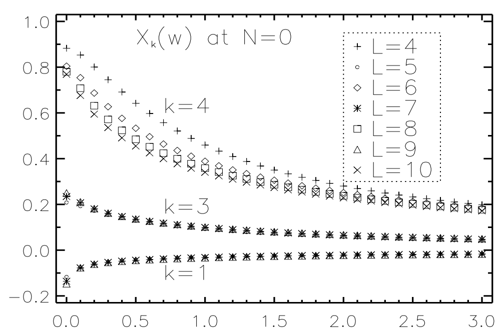

FIG. 2. Measures of the k-leg exponents $X_k = 2h_k$ vs. w for $N = 0$.

A useful check of our analysis is obtained by modifying the boundary conditions, in a similar way as in Ref. [8]. If we make the Fermi components of the ϕ fields obey antiperiodic boundary conditions, this will give a non-contractible (winding) loop the usual weight $w_{\rm e} = m - 2n = N$ if its winding number is even, and $w_{\rm o} = m + 2n = N + 4n$ if it is odd. In the Goldstone phase, this will give the n pairs of symplectic fermions the antiperiodic boundary condition, while leaving the bosonic scalar fields periodic. In the formula for $f(L)$, c will be replaced by $c - 24h_{\rm tw}$, where $h_{\rm tw}$ is the lowest conformal weight in the sector with boundary conditions twisted as described. We expect $c - 24h_{\rm tw} = m + n - 1$, which can be understood as the contribution of the twist operator of weight $h = -1/8$ for each symplectic fermion pair. Thus we expect

$$c - 24h_{\rm tw} = N - 1 + 3n = \frac{3w_o + w_e}{4} - 1, \qquad (4)$$

for general N, n. This appears to be confirmed numerically for $N = w_{\rm e} = 0$ and several values of $w_{\rm o}$ in Fig. 3. Notice how this differs from the usual loop gas behavior ($w = 0$), where $c - 24h_{\rm tw} = 1 - 6\arccos^2(\frac{w_{\rm o}}{2})/(\pi^2 g)$ [16].

For $N < -2$, we believe that the dense-loops phase is massive, and hence flows to the generic Goldstone phase only when w is greater than a positive critical value.

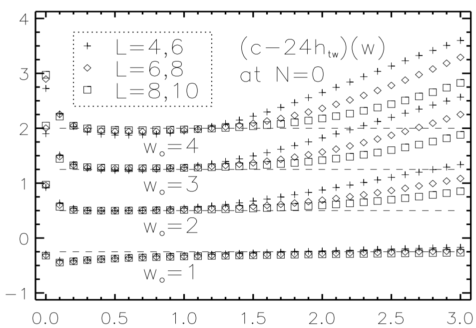

FIG. 3. Measures of $c - 24h_{\rm tw}$ vs. w for $N = 0$. Dashed lines show the prediction of Eq. (4).

Finally, it is important that a version of the OSp$(m|2n)$-invariant model on the square lattice is integrable at $w = (2 - N)/4$ and can be solved by the Bethe ansatz [5]. The model is massless for $N < 2$, and there is strong evidence that the central charge is $c = N - 1$, and that the lowest scaling dimension, other than that of the identity operator, is zero; the authors conjecture that these are the exact values, but do not identify the phase with the Goldstone phase.

To conclude, we have argued that crossing of loops is a relevant perturbation in the dense-loops low-T phase for $-2 < N < 2$, and that the long-distance behavior is governed by the Goldstone phase instead, by analytical, numerical, and exact approaches.

This work was supported by the NSF under grant no. DMR-98-18259 (NR), and by the DOE (HS).

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